Statistical Analysis of Spectral Regrowth in Wireless Nonlinear Circuits

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Abstract—**The output power spectrum of a modulated carrier applied to a nonlinear wireless circuit is analyzed using the statistical properties of the signal and a complex power series behavioral model of the nonlinear circuit. Nonlinear analysis of signals with known statistical moment properties results in reduced order expressions for the output power spectrum in terms of the autocorrelation function of the input signal and the power series coefficients. Comparisons are presented for several limiting amplifier models and between measured versus predicted adjacent channel power rejection (ACPR) values for a microwave amplifier.**

I. Introduction

Modern wireless communication systems utilize digital modulation techniques employing complex coding schemes to maximize the channel throughput for the available capacity. Design of wireless circuits relies on characterization of the channel distortion based on sinusoidal two-tone modulation which does not accurately represent the amplitude fluctuations of the modulation. However, time-domain or mixed time-frequency domain simulation of nonlinear circuits is costly in terms of simulation time required to verify performance over extreme operating conditions. An alternative approach is to use a behavioral model of the circuit combined with spectral analysis of the output of the model. The output power spectrum can be derived by moment calculations if the statistical moment properties of the signal are known. Gaussian signals have well known statistical properties and are closely related to noise problems in wireless circuits and are also associated with CDMA systems when several users or code channels are utilized at the same time.

This paper presents a statistical analysis of the output autocorrelation function and power spectrum when the input to a nonlinear circuit is either a real or complex Gaussian random process. The analysis is based on a modulated carrier passed through a complex power series behavioral model of a wireless nonlinear circuit. The output power spectrum is obtained from the Fourier transformation of the output autocorrelation function. Spectral, ACPR, and gain compression analysis results are compared for five limiter amplifier models. The statistical formulation results in simpler expressions for the output autocorrelation function and power spectrum containing significantly fewer spectral components than the general time-average autocorrelation formulation. Finally, the complex gain characteristic of a CDMA amplifier is measured and modeled as a complex power series. The model is used to calculate the output power spectrum when a complex Gaussian input signal is applied to the circuit. Measured and predicted ACPR results are compared and shown to be in excellent agreement.

II. Limiter Amplifier Models

The hyperbolic tangent function is a convenient nonlinear limiting function that also represents the large-signal response of a bipolar or heterojunction bipolar transistor differential pair amplifier

$$v_o = L \tanh\left(\frac{g}{L} v_{in}\right) \tag{1}$$

where g is the linear gain and L is the limit value of the output signal. One drawback of the hyperbolic tangent function is that the "sharpness" of the transition from the linear to the limiting characteristic of the model is fixed in relation to the gain and cannot be adjusted without introducing additional parameters.

A popular behavioral limiter model which permits independent control of gain, limiting value, and the sharpness of the transition characteristic is the Cann model [1] given by

$$v_o = \frac{g v_{in}}{\left[1 + \left(\frac{g}{L}|v_{in}|\right)^s\right]^{\frac{1}{s}}} \tag{2}$$

where g is the linear gain, L is the limit value of the output signal, and s controls the sharpness of the transition from linear to limiting.

A plot of the large signal carrier transfer characteristic of the hyperbolic tangent and Cann limiter models are shown in Fig. 1. The carrier transfer characteristic is the large-signal transfer function mapping a sinusoidal input to the first harmonic of the output. The first harmonic response is essentially the first coefficient of a Fourier series expansion which limits to $4L/\pi$ as can be seen in the plots from Fig. 1.

Each of the large signal carrier transfer characteristic curves were fitted to a power series using a least mean squared fit to the data. The output signal is obtained by using a binomial expansion of the power series model and a modulated carrier, described by a complex envelope representation, and can be shown to be [2]

$$\tilde{G}_{\omega_c}\left[\tilde{z}(t)\right] = \sum_{n=0}^{\frac{N-1}{2}} \frac{\tilde{a}_{2n+1}}{2^{2n}} \binom{2n+1}{n+1} \tilde{z}(t)^{n+1}\left[\tilde{z}^{*}(t)\right]^{n} . \quad (3)$$

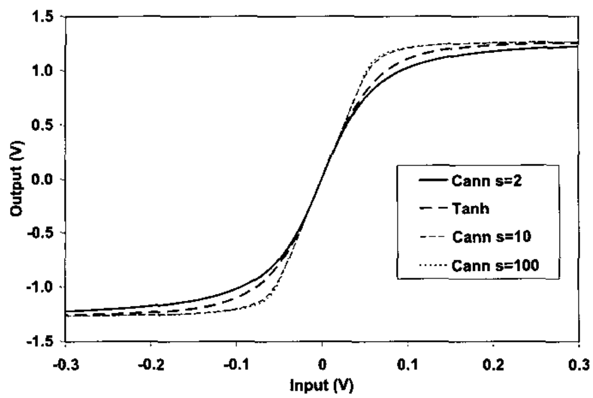

Fig. 1 Carrier transfer characteristic for limiter models.

III. Statistical Analysis of Spectral Regrowth

A bridging principle between the time and statistical domains is equivalence of statistical and time averaged moments for signals that are ergodic in regards to the autocorrelation function. Ergodicity refers to a property where the ensemble averages of random process converge to the time average of any realization of the process, in the limit, as t goes to infinity, i.e.

$$\tilde{R}_{zz}(\tau) = E\left[\tilde{z}(t)\tilde{z}^{*}(t+\tau)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{z}(t)\tilde{z}^{*}(t+\tau)dt . \quad (4)$$

Moments for random processes can be formulated from known properties of the process, but evaluation of the moments from the statistical description may be mathematically difficult. In those cases, the moments are evaluated by substituting time average autocorrelation functions for the statistical moments. There is a net reduction in complexity when the moment formulation leads to fewer autocorrelation terms than the general time average autocorrelation function formulation.

A. Transformation of a Real Gaussian Process

First consider the case where a carrier frequency is modulated by a real random process, $x(t)$,

$$w(t) = \frac{1}{2}x(t)e^{j\omega_c t} + \frac{1}{2}x(t)e^{-j\omega_c t} \quad (5)$$

where $\omega_c = 2\pi f_c$ and f_c is the carrier frequency. The modulated carrier is applied to the input of a nonlinear circuit represented by the complex power series model from (3)

$$\tilde{G}_{\omega_c}\left[x(t)\right] = \sum_{n=0}^{(N-1)/2} \frac{\tilde{a}_{2n+1}}{2^{2n}} \binom{2n+1}{n+1} x(t)^{2n+1} . \quad (6)$$

The output autocorrelation function is found by taking the expectation of the output signal

$$\begin{aligned}\tilde{R}_{gg}(\tau) &= E\left[\tilde{G}_{\omega_c}(x_1)\tilde{G}_{\omega_c}^{*}(x_2)\right] \\ &= \sum_{n=0}^{\frac{N-1}{2}}\sum_{m=0}^{\frac{N-1}{2}} \frac{\tilde{a}_{2n+1}\tilde{a}_{2m+1}^{*}}{2^{2(n+m)}} \binom{2n+1}{n+1}\binom{2m+1}{m+1} E\left[x_1^{2n+1}\,x_2^{2m+1}\right].\end{aligned} \quad (7)$$

The expectation can only be evaluated if the moments of the random variable are known. For the case of a zero mean real Gaussian process the moments are given by [3]

$$E\left[x_1x_2...x_s\right] = \begin{cases} 0 & , s \text{ odd} \\ \sum\limits_{\substack{\text{all two}\\ \text{pair}}} \left\{E\left[x_1x_2\right]E\left[x_3x_4\right]...E\left[x_{s-1}x_s\right]\right\} & , s \text{ even}. \end{cases} \quad (8)$$

The output autocorrelation function is obtained by calculating each of the expectations in (7) using (8) then collecting terms of like powers. After computing several of the moments and collecting like power terms, it can be shown that the autocorrelation function terms follow the following pattern

$$\tilde{R}_{gg}^{2k+1}(\tau) = \left|\sum_{n=k}^{\frac{N-1}{2}} \frac{\tilde{a}_{2n+1}(2n+1)!(2n+1)!}{2^{3n-k}(n-k)!(n+1)!n!} R_{xo}^{n-k}\right|^{2} \frac{\tilde{R}_{xx}^{2k+1}(\tau)}{(2k+1)!} . \quad (9)$$

The output autocorrelation function expression for a carrier modulated by a real Gaussian random process passed through a bandpass nonlinearity is the sum of nonlinear autocorrelation terms. Thus the output autocorrelation function is a sum of $(N-1)/2$ terms for an N*th* odd order power series where each autocorrelation term is weighted by a sum of DC powers from other nonlinear terms of equal and higher order. The output autocorrelation function is a closed-form expression in terms of autocorrelation of the input signal and the sum of the input power weighted by the power series coefficients. The output power spectrum is the Fourier transform of the autocorrelation function

$$\tilde{S}_{gg}(f) = \sum_{k=0}^{\frac{N-1}{2}} \tilde{S}_{gg}^{2k+1}(f) \quad (10)$$

where

$$\tilde{S}_{gg}^{2k+1}(f) = \left|\sum_{n=k}^{\frac{N-1}{2}} \frac{\tilde{a}_{2n+1}(2n+1)!(2n+1)!}{2^{3n-k}(n-k)!(n+1)!n!} R_{zo}^{n-k}\right|^{2} \frac{F\left\{\tilde{R}_{xx}^{2k+1}(\tau)\right\}}{(2k+1)!} \quad (11)$$

and $F\left\{\tilde{R}_{xx}^{2k+1}(\tau)\right\}$ is the Fourier transform of $\tilde{R}_{xx}^{2k+1}(\tau)$. Use of the moment theorem yielded a closed form expression for the output spectrum with $(N-1)/2$ unique spectral terms.

B. Transformation of a Complex Gaussian Process

The problem is more complicated when extended to the case where a carrier is modulated by a complex random process. For this case, the output autocorrelation function is formulated by taking the expectation of the output of the nonlinear amplifier with a modulated carrier applied to the input from (3)

$$\begin{aligned}\tilde{R}_{gg}(\tau) &= E\left[\tilde{G}_{\omega_c}(\tilde{z}_1)\tilde{G}^*_{\omega_c}(\tilde{z}_2)\right] \\ &= \sum_{n=0}^{\frac{N-1}{2}}\sum_{m=0}^{\frac{N-1}{2}} \frac{\tilde{a}_{2n+1}\tilde{a}^*_{2m+1}}{2^{2(n+m)}}\binom{2n+1}{n+1}\binom{2m+1}{m+1}\times \\ &\qquad E\left[\tilde{z}_1^{n+1}\left(\tilde{z}_1^*\right)^n\left(\tilde{z}_2^*\right)^{m+1}\tilde{z}_2^m\right].\end{aligned} \tag{12}$$

Thus the power series expansion and resulting cross terms in the autocorrelation function yield the following expression for the expectation term

$$\begin{aligned}\tilde{R}_{z_{2n+1}z_{2m+1}}(\tau) &= \frac{\tilde{a}_{2n+1}\tilde{a}^*_{2m+1}}{2^{2(n+m)}}\binom{2n+1}{n+1}\binom{2m+1}{m+1}\times \\ &\qquad E\left[\tilde{z}_1^{n+1}\left(\tilde{z}_1^*\right)^n\tilde{z}_2^m\left(\tilde{z}_2^*\right)^{m+1}\right].\end{aligned} \tag{13}$$

Evaluation of the expectation function is more complicated for this case due to the complex conjugate terms generated by the envelope of the carrier. Expanding $\tilde{R}_{gg}(\tau)$ results in many algebraically intensive moment manipulations involving $x(t)$ and $y(t)$. Fortunately, a previous result for the moments of complex Gaussian random variables [4] can be used to calculate each of the terms in this expression, namely

$$E\left[\tilde{z}_1\tilde{z}_2\ldots\tilde{z}_s\tilde{z}_1^*\tilde{z}_2^*\ldots\tilde{z}_t^*\right] = \begin{cases} 0 & , s \neq t \\ \sum_{\pi} E\left[\tilde{z}_{\pi(1)}\tilde{z}_1^*\right]E\left[\tilde{z}_{\pi(2)}\tilde{z}_2^*\right]\ldots E\left[\tilde{z}_{\pi(s)}\tilde{z}_t^*\right] & , s = t \end{cases} \tag{14}$$

where π is a permutation of the set of integers $\{1,2,\ldots,s,\ldots,t\}$, and $\{\tilde{z}_i, i=1,2,\ldots,s,\ldots,t\}$ denotes a set of complex independent Gaussian random variables.

The output autocorrelation function is obtained by calculating each of the expectations in (13) using (14) then collecting terms of like powers. After computing several of the moments and collecting like power terms, it can be shown that the autocorrelation function terms follow the following pattern

$$\tilde{R}_{gg}^{2k+1}(\tau) = \left|\sum_{n=k}^{\frac{N-1}{2}} \frac{\tilde{a}_{2n+1}}{2^{2n}}\frac{(2n+1)!}{(n-k)!}R_{zo}^{n-k}\right|^2 \frac{\tilde{R}_{zz}^{k+1}(\tau)\left[\tilde{R}^*_{zz}(\tau)\right]^k}{k!(k+1)!}. \tag{15}$$

Thus similarly to the real bandpass Gaussian case the output autocorrelation function is a sum of $(N-1)/2$ terms for an N*th* odd-order power series; although, the autocorrelation terms are products of the autocorrelation and its complex conjugate. The output autocorrelation function is a closed form expression in terms of autocorrelation of the input signal and the sum of the input power weighted by the power series coefficients. The output power spectrum is the Fourier transform of the autocorrelation function

$$\tilde{S}_{gg}(f) = \sum_{k=0}^{\frac{N-1}{2}} \tilde{S}_{gg}^{2k+1}(f) \tag{16}$$

where

$$\tilde{S}_{gg}^{2k+1}(f) = \left|\sum_{n=k}^{\frac{N-1}{2}} \frac{\tilde{a}_{2n+1}}{2^{2n}}\frac{(2n+1)!}{(n-k)!}R_{zo}^{n-k}\right|^2 \frac{F\left\{\tilde{R}_{zz}^{k+1}(\tau)\left[\tilde{R}^*_{zz}(\tau)\right]^k\right\}}{k!(k+1)!} \tag{17}$$

and $F\left\{\tilde{R}_{zz}^{k+1}(\tau)\left[\tilde{R}^*_{zz}(\tau)\right]^k\right\}$ is the Fourier transform.

Similarly to the real bandpass Gaussian case, use of the moment theorem yielded a closed form expression for the output spectrum in terms of the input autocorrelation function with $(N-1)/2$ distinct spectral terms.

C. Spectral Results

A composite plot of the output power spectrum, using the Gaussian moment results from (16), for each of the limiter models with a complex Gaussian input signal applied and an output power of 2 dBm is shown in Fig. 2. The out-of-band distortion is highest for the softer limiter models like the Cann s=2 and hyperbolic tangent models, and lowest for models with a sharper nonlinear transition for the same output power. The spectrum of the input signal limits the measurable output power spectrum at far offsets to the carrier which is why the far out spectrums converge to nearly the same value at offsets greater than 4 MHz.

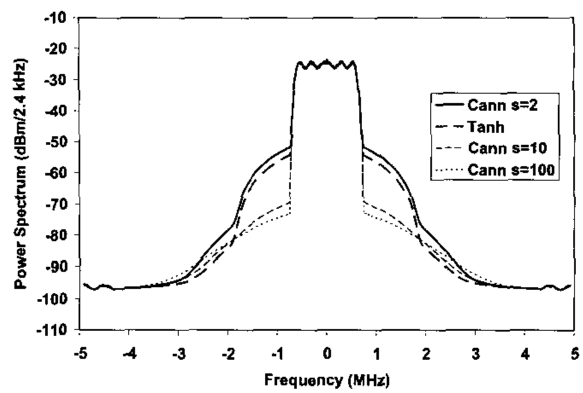

Fig. 2 Power spectrum at 2 dBm with complex Gaussian input signal.

The ACPR versus output power plots for each of the limiter models with real and complex Gaussian input signals are shown in Fig. 3 and Fig. 4 respectively for a distortion offset of 885 KHz. There are differences between ACPR produced by the different models similar to the differences observed with a CDMA input signal. The triplet multi-tanh, Cann s=10, and Cann s=100 models exhibit a 2:1 slope at low output power level; however, there are noticeable notches in the ACPR response in the 0 dBm to 6 dBm output power range.

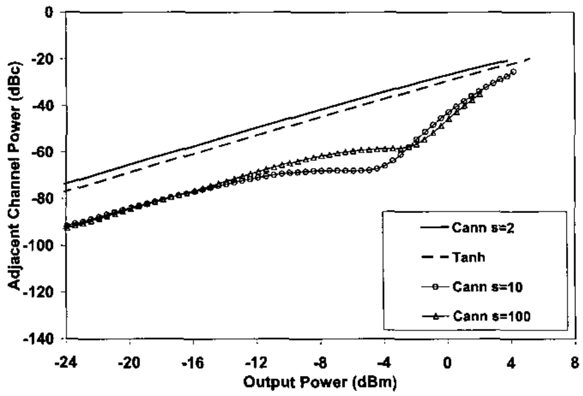

Fig. 3 Real Gaussian ACPR sweep.

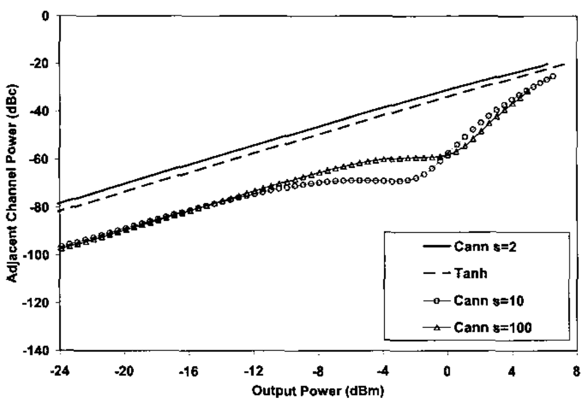

Fig. 4 Complex Gaussian ACPR sweep.

IV. Measurements

The complex Gaussian moment method was used to calculate the output power spectrum and ACPR of an integrated RF amplifier with complex Gaussian input signal. The device under test (DUT) is a 835 MHz CDMA/AMPS driver amplifier device fabricated using a GaAs MESFET technology [5]. A vector network analyzer, with a built in power sweep function, was used to measure the AM-AM and AM-PM over an input power range of -25 dBm to -2 dBm. The measured AM-AM, AM-PM characteristics were fit to a complex power series of odd order N=13 using a least squares solution.

A carrier modulated with a complex Gaussian signal is applied to the amplifier circuit and the output distortion measured using a spectrum analyzer. ACPR is the ratio, in decibels, of the distortion power, in a 30 kHz bandwidth offset by ±885 kHz, and the desired channel power, in a 1.23MHz bandwidth. The ACPR measurements along with the predicted ACPR by the complex Gaussian formulation are shown in Fig. 5. The measured and predicted ACPR are in good agreement below an output power level of 11 dBm. The complex Gaussian moment formulation deviates from the measured data because of the limited dynamic range of the complex power series model of the nonlinear amplifier.

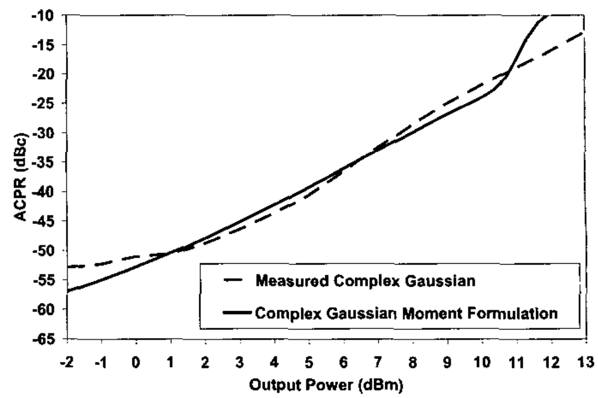

Fig. 5 Measured and calculated ACPR for complex Gaussian input signal.

V. Conclusion

A statistical method for analyzing the power spectrum of modulated carriers passed through a nonlinear wireless circuit was presented. The method is based on a transformation of the statistical properties of the modulated carrier passed through a complex power series model of the nonlinear amplifier. The second order moments of the nonlinear terms are calculated and combined leading to a closed form expression of the output autocorrelation in terms of the autocorrelation function of the input signal. The statistical formulation yields $(N-1)/2$ spectral terms for real and complex Gaussian modulation of the carrier. The analysis technique was applied to limiter amplifier models and demonstrated excellent agreement when compared against measured data from a CDMA amplifier.

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